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Review Article

(FG,σ)- Purity and Semi-simple Modules

Ashok Kumar Pandey*

Department of Mathematics, Ewing Christian Post Graduate College (An Autonomous College of University of Allahabad, Prayagraj), Allahabad, India

***Corresponding Author:** Ashok Kumar Pandey, Department of Mathematics, Ewing Christian Post Graduate College (An Autonomous College of University of Allahabad, Prayagraj), Allahabad, India.

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Abstract

The torsion sub-module of A⊆M is denoted by σ(A). Since it was proved by Walker [18] that the class of I- pure (J- copure) sequences form a proper class whenever I(J) is closed under homomorphic images (sub-modules) of a R- module M and if I(J) is closed under factors (sub-modules) then for any I- pure (J- copure) sequence E:0⟶A ⟶B ⟶C ⟶0 if E ∈π^(-1) (I) (E ∈i^(-1) (I)) and hence in this case Walker's I- purity (J- copurity) coincides with the earlier notion of purity. We also study about class of R-modules dual to the modules of B. A sequence E:0⟶A ⟶B ⟶C ⟶0 is I- pure (J- copure) if and only if given C^'≤C∈ I, there existsB'≤B such that B^'≅C' and A∩B^'=0 ;we consider another notion of purity stronger than the Cohn's purity [13]. If FG denotes the class of all finitely generated R-modules, since, this class is closed under factors. We shall try to give some characterizations of FG-purity and to determine its relationship with the FG-flat modules. We relativist this concept and also relate it with that of finite projectivity of Azumaya [10] with respect to a torsion theory and to study the inter-relationship between these concepts. We also try to consider finite σ-projectivity or (FG,σ)- pure flatness, cyclically σ- pure projectivity and cyclically σ- pure flatness, the concept of locally σ- projectivity and locally σ- splitness and study its inter-relationship with (FG,σ)- purity and semi-simple module.

Keywords: R- Modules; (FG,σ)- Purity; σ- Pure Projective; R-Modules; I- Pure (J- copure; FG-flat Modules; Cyclically σ- Pure Projectivity; σ- Pure Infectivity; Locally σ- Splitness; Semi-Simple Module. Subject classification: 16D99

Introduction

The notion of purity plays an abecedarian part in the theory of abelian groups as well as in module categories. We say that an R- module M is absolutely pure, (respectively regular, flat) with reference to the purity if any short exact sequence with M as the first (respectively second, third) position is pure in the given sense. Now we take a free presentation of N where N is a right R- module and $\oplus_J R \xrightarrow{\mu} \oplus_I R \longrightarrow N \longrightarrow 0$.

We take all the sub-matrices associated with μ are of the column finite matrix. The class of all co-kernels of the right R- maps between ⊕J R and ⊕I R convinced by these sub-matrices is expressed by ℘ (N). Now we take allrow finite sub-matrices of the matrix and take co-kernels of all left R- maps between ⊕I R and ⊕J R induced by these sub-matrices and this class of left R − modules is denoted by f(N). An accurate sequence E: 0 ⟶ A ⟶ B ⟶ C ⟶ 0 is called τ − pure (𝔉- copure) if any torsion (torsion free) module is projective (injective) relative to it. Since τ(𝔉) is closed under factors (sub-modules). In this situation Walker's criterion of Co-purity is applicable. The notation of a R − module M is τ −pure projective (𝔉- copure injective) if and only if $Pext_\tau(M, A) = 0$ ($Pext_𝔉(A, M) = 0$) for all A ⊆ M. Since, $Pext_\tau(T, A) = 0$ for all T ∈ τ.

The torsion sub-module of A⊆ M is denoted by σ (A). It's proved by Walker that the class of f − pure (𝒥 − copure) sequence form a proper class when f(𝒥) is closed under homomorphism images (sub-modules) of an R- module M and if f(𝒥) is closed under factors (sub-modules) then for any f − pure (𝒥 − copure)sequence

E: $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ and $E \in \pi^{-1}(f)$ $(E \in i^{-1}(f))$.Therefore, in this case Walker's f – purity ($\mathcal{J}$ – copurity) coincides with the previous notion of purity. We also study about class of R –modules dual to the modules of $\mathfrak{B}$. A sequence E: $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is f – pure ($\mathcal{J}$ – copure) if and only if given $C' \le C \in f$, there exists $B' \le B$ such that $B' \cong C'$ and $A \cap B' = 0$; we consider another notion of purity stronger than the Cohn's purity. However, since, this class is closed under factors, if $\mathcal{FG}$ denotes the class of all finitely generated R- modules.

We shall try to give some characterizations of $\mathcal{FG}$ –purity and to determine its relationship with the $\mathcal{FG}$ –flat modules.

An R – module M is called locally projective if given a map g: $M \longrightarrow B$ and a finitely generated sub-module F of M, if there exists a map g': $M \longrightarrow A$ such that $(\pi o g')|F = g|F$, that is

$$\begin{array}{ccccc} 0 & \longrightarrow & F & \longrightarrow & M \\ & & \downarrow & & \\ A & \xrightarrow{\pi} & B & \longrightarrow & 0 \end{array}$$

We know that all locally projective modules are flat and this class lies strictly between flat and projective modules. We relate these concepts to $\mathcal{FG}$ – purity which is same as finite splitness with respect to a hereditary torsion theory which is given by a connection of $(\mathcal{FG}, \sigma)$ – purity with $\tau\mathcal{F}$ – purity (torsion purity) of a left exact torsion radical σ. We also relativize the concept of finite (cyclic) σ – extension and finitely (cyclically) σ –splitness. In this present paper we related the concept of $\mathcal{FG}$ - purity and σ – injectivity and σ –projectivity of R – modules. We also, relate the concept of $\mathcal{FG}$ - purity and semi-simple modules. We observe that the torsion σ – purity of Bhattacharya and Choudhury [11] reduces to usual purity, $(\mathcal{FG}, \sigma)$ – splitness and cyclic σ – purity becomes purity relative to cyclic modules that is singly (cyclically) σ – pure. We also give the results of the characterization of a Noetherian like condition on the torsion theory.

τ_1- Purity coincides with the usual purity (Cohn purity), that to an abelian groups only. In this paper we also try to develop the theory of σ –purity relative to a torsion theory (τ, τ_1) which is weaker than τ –purity but it gives the generalization of usual purity (Cohn purity) [13] and also gives a σ –generalization of regular modules.

Definition

- A R – module M is said to be cyclic if and only if there exists an element $m_0 \in M$ such that $M = Rm_0$.
- A R – module M is said to be finitely generated if and only if there exists a finite generating set X of M.
- A left R – module M is said to finitely co-generated if and only if for each set $\{Ui \mid i \in I\}$ of submodules Ui of M with $\cap_{i\in I}$ Ui = 0, there exists a finite subset $\{Ui \mid i \in I0\}$ that is $I0 \subset I$ and I0 is finite with $\cap_{i\in I}$ Ui = 0. In other words we can say A module M is said to be finitely co-generated if it is co- generated by the family $\{E(S_{i\in I})\}$ finitely. That's $E(M) = \bigoplus_{i=1}^{n} E(Si)$ where Si∈I, simple modules are not inescapably non- isomorphic.
- An R – module M is said to be co-cyclic if it is contained in E(S) for some simple module S, where E(S) is a family of co- generators for each R module M.
- In the commutative illustration $\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ M & \to & N \end{array}$ Where f: $A \longrightarrow B$; φ: $M \longrightarrow N$, μ: $A \longrightarrow M$ and g: $B \longrightarrow N$ aremaps. The pair (φ, g) is said to be the push out of the pair (μ, f) if and only if for every pair (φ', g') with φ': $M \longrightarrow X$, g': $B \longrightarrow X$ and $(\varphi' o\, \mu) = (g' of)$, there exists a unique map σ: $N \longrightarrow X$ similar that $(\sigma og) = g'$.
- The pair (ϕ, f) is said to be the pullback of the pair (ψ, g) if and only if for every pair (ϕ', f') with ϕ': $Y \to M$, f': $Y \to B$ and $(\psi o \phi') = (gof')$, there exists a unique map τ: $Y \to A$ Similarly $(fo\tau) = f'$ and $(\phi o \tau) = \phi'$.
- A R – module M is said to be finitely presented if there's an exact sequence $M_1 \longrightarrow M_0 \longrightarrow M \longrightarrow$ where M_0 and M_1 are independent modules with finite bases.
- Let R be a ring and M is a left R – module, then M is said to flat if for every exact sequence $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ and the converted sequence $0 \longrightarrow M \otimes R\ N' \longrightarrow M \otimes R\ N \longrightarrow M \otimes R\ N'' \longrightarrow 0$ is exact.
- A ring R is hereditary if and only if every ideal is a projective module.
- If M be a R –module, the sum of all simple sub-modules of M is called the socle of M and it is denoted by $s(M) = \{x \in M \mid Ann(x)$ is a finite intersection of maximal right ideals$\}$. That is if $x \in s(M)$, then xA isa direct sum of a finite number of simple modules where A is a semi-simple ring.

- A non- zero module S is said to be simple if it has on submodules other than {0} and S. A module is saidto be semi-simple if it is a sum of simple sub-modules.
- A torsion theory is a pair $(f, \mathfrak{F})$ of classes of modules satisfying:
  - $\mathrm{Hom}(T, F) = 0, \forall\, T \in f$ and $F \in \mathfrak{F}$
  - $\mathrm{Hom}(L, F) = 0, \forall\, F \in \mathfrak{F} \Rightarrow L \in f$
  - $\mathrm{Hom}(T, N) = 0, \forall\, T \in f \Rightarrow N \in \mathfrak{F}$
- The classes $\mathfrak{F}$ and f are known as torsion free and torsion classes associated with a torsion theory$(f, \mathfrak{F})$. A torsion theory $(f, \mathfrak{F})$ is said to be hereditary if and only if f is closed under homomorphism images, direct sums, extensions and submodules. Also, $\mathfrak{F}$ is closed under submodules, direct products, extensions and injective envelopes.
- A left R – module P is said to be σ – pure projective module if it's projective to relative to every σ – pure epimorphism. That is given any σ – pure exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ and a homomorphism $f: P \longrightarrow C$, there exists a map $h: P \longrightarrow B$ Corresponding that poh = f where $p: B \longrightarrow C$be an on to homomorphism.
- A left R – module Q is said to be finitely σ – pure injective if it is $(\mathcal{FG})$, σ- pure in every pure extension of Q, that is if $0 \longrightarrow Q \longrightarrow Q' \longrightarrow Q'' \longrightarrow 0$ is a pure exact sequence then it's $(\mathcal{FG})$, σ- pure also.Similarly,Q is said to be cyclically σ – pure injective if it is cyclically σ – pure in every pure extension of it.
- A sub-module A of an R-module B is called closed if B|A is torsion free and it is called dense if B|A is torsion. Any closed submodule A of an R-module B is τ –pure.
- A sub-module $A \subseteq M$ is called f – essential if it intersects every torsion sub module of M.

Definition 1.1

A sub-module A of an R-module B is called closed if B|A is torsion free and it is called dense if B|A is torsion. Any unrestricted sub module A of an R-module B is τ –pure.

Definition 1.2

Given a class of modules τ , a sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is known τ – pure if A is a direct summand of D whenever $A \subseteq D \subseteq B$ and $D|A \in \tau$.

Walker proved that the class of τ – pure sequences form a correct class whenever τ is closed under homomorphism of a R – module M and if τ is closed under factors then for any τ – pure sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0, E \in \pi^{-1}(\tau)$ and hence in this case Walker's τ – purity coincides with the earlier notion.

Proposition 1.3

If τ is closed under factors also a sequence $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is τ – pure if and only if given $C' \le C \in \tau$ there exists $B' \le B$ similar that $B' \cong C'$ and $A \cap B' = 0$.

Definition 1.4

We say that a sub module A is (μ, σ) –pure in an R – module B, if any system of linear equation $\sum r_{ij} x_j = ai$ given by the row finite matrix μ in A, whenever solvable in B in the form $x_j = bi$ for which there are left ideals $D_i \in D$ where D is the Gabriel filter [6] of dense left ideals corresponding to the left exact torsion radical σ, such that $D_j b_j \in A$. The system is also solvable in A that is there are $a_j' \in A$, with $\sum r_{ij} a' = a_i$ for each $i \in I$ and $j \in J$; this exactly means that given vectors $(b_j) \in \prod_j B$ and $(a_i) \in \prod_I A$ and $\mu(b_j) = a_i$ With $D_j b_j \in A$ for some $D_i \in D$, there exists $(a_j') \in \prod_I A$ such that $\mu(a_j') = ai$ where the vector $\mu(a_j')$ is obtained by matrix product of the row finite matrix μ and column vector(a_j'). We may rephrase the above condition that a sub module A is (μ, σ) –pure in a R – module B or that B is a (μ, σ) –pure extension of A as follows.

We view μ as mapping $\prod_J B$ to $\prod_I B$ by left matrix multiplication. Then we have: Theorem 1.5

A sub module A is (μ, σ) –pure in B if and only if $\mu[\prod_j B] \cap \prod_I A \subseteq \mu[\prod_I A]$ whenever B_j' are sub modules of B containing A such that A is dense in B_j.

Proof

Any element of the left hand side is of the form $(a_i)I = \mu((b_j)J) = \sum r_{ij} b_j$ and A is dense in B_j means $B_j|A$ is torsion and hence for each element$(b_j + A) \in B_j|A$, there exists $D_j \in D$ such that $D_j(b_j + A) = 0$ that is $D_j(b_j) \subseteq A$.

The following result corelates (μ, σ) –purity with (M,σ) –purity.

Proposition 1.6

Let $\mu = (R_{ij})$ be a row finite (I × J) matrix where I and J are arbitrary sets. Again a sub module A is (μ, σ) –pure in a module B if and only if the sequence $0 \longrightarrow A \longrightarrow B \longrightarrow B|A \longrightarrow 0$; is (M, σ) –pure where $\oplus iR \xrightarrow{\mu} \oplus jR \longrightarrow M \longrightarrow 0$ is exact with μ' given by the matrix μ.

Definition 1.7

A sub module A is τ – pure in a R – module M if and only if given a torsion sub module C of M|A, there exists asubmodule B of M similarly $B \cong C$ and $A \cap B = 0$.

Definition 1.8

A sub module A of an R – module M is called μ – pure in M where $\mu = (x_{ij})$ if whenever the system of linear equations $\sum r_{ij} x_j = a_i$, $i \in I$ where ai $\in$ A with $D_j(x_j) \subseteq A$ for some $D_j \in D$, it associated with Gabriel filter for left dense ideals, is solvable in M,that is it is solvable in A.

Proposition 1.9

A sub module A of a R – module is σ – pure in M if and only if A is (Cohn)– pure [11] in the closure of A in M.

Proposition 1.10

A sub module A of a R – module M is μ –pure in M if and only if A is M – pure in the closure of A in M where

$\oplus iR \xrightarrow{\mu`} \oplus jR \longrightarrow M \longrightarrow 0$ is exact.

Proof

The closure $\bar{A}$ of A is defined by $\bar{A}\,|A = \sigma(M|A)$. Still, again by Azumaya [10], A is μ –pure in $\bar{A}$, if A is μ –pure in $\bar{A}$ Likewise the presented a finite system of linear equations in a finite number of variables $\sum r_{ij} b_j = a_i$,; $i \in I$ where $a_i \in A$ with $D_j(x_j) \subseteq A$ for some $Dj \in D$, $mj + A \in \sigma(M|A) = \bar{A}\,|A$. Hence, $mj \in \bar{A}$ as A is pure in $\bar{A}$ there exists $a_j' \in A$ similarly that $\sum r_{ij} a'_j = a_i$, and the system is solvable in A. Conversely, if the given a finite system of linear equations in a finite number of variables $\sum r_{ij} m_j = a_i$, $i \in I$ with $a_i \in A$ and $m_j \in \bar{A}$ then, $m_j + A \in \bar{A}\,|\,A = \sigma(M|A)$ there is $D_j \in D$, $D_j(m_j + A) = 0$ that is $D_j(b_j) \subseteq A$ and hence the system is solvable in A and so, A is pure in $\bar{A}$Hence A is μ –pur in $\bar{A}$ by Azumaya [10] proposition (1).

Definition 1.11

- A R – module C is said to be σ – flat if a sub module A is σ – pure in an R – module B whenever $C \cong B|A$.
- A sub module A of a R – module B is said to be (μ, σ) – pure if and only if $A \subseteq \bar{A}_i$ s μ – pure.
- We call a sub module A of a R- module B, τ –essential if it intersects every torsion sub module of B.

Proposition 1.12

Every torsion free module is σ – flat and every torsion σ – flat module is flat. Also, every flat module is σ – flatof course.

Proposition 1.13

A sub module A is closed in B if and only if A is τ –pure and τ –essential in B.

Proof

If Ais closed in B then A is τ –pure. Suppose that $A \cap B1 = \{0\}$ for some $B1 \subseteq B$ and $B1 \in \tau$. But $B1 \subseteq \sigma(B)$ and $\sigma(B) = \cap C$, where $C \subseteq B$ and $B/C \in \tau$ and hence $B_1 \subseteq A$ because $B/A \in \tau$. Thus A is τ –essential.

Conversely, if A is τ –pure and τ –essential in B, if B/A has any torsion sub module C then $C \approx B \subseteq B$ and $A \cap B_1 = \{0\}$ for some $B_1 \subseteq B$ and $B_1 \in \tau$, thus A cannot be τ –essential. Hence, $B/A \in \tau$

Now we give the inter-relationship with (FG, σ) – purity and semi-simple module.

Proposition 1.14

The exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is τ – pure exact if and only if $0 \longrightarrow \sigma(A) \longrightarrow \sigma(B) \longrightarrow \sigma(C) \longrightarrow 0$ is a split exact sequence where the maps are restrictions of the above sequence.

Proof

Suppose that the sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is τ – pure exact. Now we complete the diagram by taking pullback of j_C: $\sigma(C) \longrightarrow C$ and π: $B \longrightarrow C$.Here, t: $K \longrightarrow \sigma(B)$; u: $\sigma(A) \longrightarrow \sigma(B)$; v: $\sigma(B) \longrightarrow \sigma(C)$; α: $\sigma(C) \rightarrow \sigma(B)$; s: $\sigma(B) \rightarrow P$.

$$
\begin{array}{ccccccccl}
 & & & & K & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & \sigma(A) & \longrightarrow & \sigma(B) & \longrightarrow & \sigma(C) & \longrightarrow 0 & \ldots\ldots\ldots\ldots (1) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & P & \underset{\lambda}{\rightarrow} & \sigma(C) & \longrightarrow 0 & \ldots\ldots\ldots\ldots (2) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow 0 & \ldots\ldots\ldots\ldots (3)
\end{array}
$$

q: $P \rightarrow B$; j_B: $\sigma(B) \longrightarrow B$, i': $A \rightarrow P$, π': $P \rightarrow \sigma(C)$; λ: $\sigma(C) \rightarrow P$, i: $A \rightarrow B$, π: $B \rightarrow C$ are the neededhomomorphism. Here s: $\sigma(B) \rightarrow P$ exists as P is a pullback. Put K = ker(v). Now vou = 0 and so, $\sigma(A) \subseteq K$. Since sequence (1) is τ –pure $\Longrightarrow$ sequence (2) is τ – Pure because τ –pure sequences form a proper class and hence (2) splits. Take λ: $\sigma(C) \rightarrow P$ such that $\pi' o\, \lambda = 1_{\sigma(C)}$. Now $\lambda(\sigma(C))$ is torsion and so there is $\alpha: \sigma(C) \to \sigma(B)$ such that $\lambda = so\alpha$. Also, $vo\alpha = \pi' o(so\alpha) = \pi' o \lambda = 1_{\sigma(C)}$ and hence, v is epic and the sequence $0 \longrightarrow K \rightleftarrows \sigma(B) \longrightarrow \sigma(C) \longrightarrow 0$ splits. But then K is an epimorphic image of σ(B) and so, it is torsion. Also, $\pi' o(sot) = 0 \Longrightarrow K \subseteq A$. Hence, $K \subseteq \sigma(A)$ and sequence (3) is separate and exact.

Again, if sequence (3) is disassociate and exact, then given $T \in \tau$, and $f: T \longrightarrow C$, $Im(f) \subseteq \sigma(C)$ and also, sequence (1) τ –pure.

$$\begin{array}{ccccccccc} & & & & T & & & & \\ & & & & \downarrow & & & & \\ 0 \longrightarrow & \sigma(A) & \longrightarrow & & \sigma(B) & \rightleftarrows & \sigma(C) & \longrightarrow 0 \ldots\ldots\ldots\ldots (4) \\ & \downarrow & & & \downarrow & & \downarrow & \\ 0 \longrightarrow & A & \longrightarrow & & B & \longrightarrow & C & \longrightarrow 0 \ldots\ldots\ldots (5) \end{array}$$

Note 1.15

If sequence (1) is τ –pure, so it's exact on sequence (1) and hence, $\sigma(A) = A \cap \sigma(B)$ and $\sigma(B)+A/A = \sigma(B/A)$.

Now we, define τ_C – purity corresponding to the class τ_C of cyclic torsion modules. This purity was firstly studied by Stenstrom [17]. More generally he started with a family ϑ of factors of a projective generator Fand he called the purity $\pi^{-1}(\vartheta)$, further he took the family $\vartheta' = \vartheta \backslash \{F\}$, and considered the relation between $\pi^{-1}(\vartheta)$ and the torsion theory generated by (ϑ′). He also established that:

- σ(M), the torsion submodule corresponding to the above torsion theory is the lowest ϑ –pure subobject of M such that each $f: P \longrightarrow M$ with $P \in \vartheta'$ factors through it.
- $L \subseteq M$ is ϑ – pure in M and contains σ(M) if and only if $\sigma(M/L) = 0$ for all $P \in \vartheta'$.

These conditions fully satisfy the case for τ_C – purity, if given the torsion theory(τ, τ_1), we take $\vartheta = \{R\} \cup$ {all cyclic torsion modules}. Yet the purity is $\pi^{-1}(\tau C)$.

Since, $\pi^{-1}(\tau C) = \pi^{-1}(\{R\} \cup \tau C)$ as R is projective and the generated torsion theory is identical as the original.

We express the set of dense left ideals by $\mathcal{D}$ that's there's left ideal I such that $R/I \in \tau C$. In this case (τ, τ_1) is heritable $\mathcal{D}$ forms a Gabriel filter or a topology Stenstrom) [17].

τ_C – Purity coincides with the purity defined by Lambek [16] in case the torsion theory is heritable.

Proposition 1.16

If (τ, τ_1) is heritable again the ensuing conditions are alternative for a sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of left R – modules.

(i) A is τ_C – pure in B.

(ii) Given $n \in \sigma(C)$, there is $m \in B$ such that $Ann(m) = Ann(n)$ and $\lambda(m) = n$.

(iii) A is pure in B in the sense of Lambek that is given $m \in B$, and $D \in \mathcal{D}$ such that $Dm \subseteq A$, there is $l \in A$ such that $D(m - l) = 0$.

Note 1.17

For the case of abelian groups and the usual torsion theory, the above purity coincides with the usual purity.

Proposition 1.18

For any class ϑ, the following statements are alternative for any R – module M:

(i) M is absolutely ϑ – pure.

(ii) M is injective module relative to any sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of left R – modules with $C \in \vartheta$.

(iii) $Ext(C, M) = 0$ for all $C \in \vartheta$.

(iv) C is $i^{-1}(M)$ – flat for all $C \in \vartheta$.

Proof: (i) ⇒ (ii)

$$\begin{array}{ccccccc} 0 \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & M & \longrightarrow & P & \longrightarrow & C & \longrightarrow 0 \end{array}$$

Since, it is given a homomorphism $A \to M$, we complete the diagram by push out. Now (ii) is ϑ – pure and hence homotopy exists, so M is injective relative to any sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of left R – modules with$C \in \vartheta$.

(ii) ⇒ (iii). Given any sequence $0 \longrightarrow M \longrightarrow P \longrightarrow C \longrightarrow 0$, in which M is injective relative to it and hence itsplits.

(iii) ⇒ (i).

$$\begin{array}{ccccccc} 0 \longrightarrow & M & \longrightarrow & P & \longrightarrow & C & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & M & \longrightarrow & A & \longrightarrow & B & \longrightarrow 0 \end{array}$$

It's that E and $C \in \vartheta$, now we complete the illustration by pullback. Now by the theory of the upper sequence splits and hence there's a homotopy and hence, the given sequence is ϑ – pure.

(ii) $\Leftrightarrow$ (iv). It is egregious.

Now dually we've M is ϑ – copure flat if and only if M is projective with respect to any sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ with $A \in \vartheta$ that is if and only if $\mathrm{Ext}(M, A) = 0$ for all $A \in \vartheta$. Now we try to specify τ –pure injective and τ –pure projective modules.

Proposition 1.19

The following statements are equivalent for any R – module M:

(i) M is τ –pure injective.

(ii) $\mathrm{Pext}_{\tau}(N, M) = 0$ for all R –modules N.

(iii) $\mathrm{Ext}(F, M) = 0$ for all $F \in \tau_1$.

(iv) M is absolutely τ_1 – pure.

(v) M is injective with respect to closed sub modules.

Proof

(i) $\Rightarrow$ (ii) Given any τ –pure sequence $0 \longrightarrow M \longrightarrow L \longrightarrow N \longrightarrow 0$, as M is τ –pure injective and so it splits. Hence $\mathrm{Ext}(F, M) = 0$ for all $F \in \tau_1$.

(i) $\Rightarrow$ (ii).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \ldots\ldots\ldots\ldots (1) \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \rightleftarrows & P & \rightleftarrows & C & \longrightarrow & 0 \end{array}$$

Suppose that sequence (1) is a τ –pure sequence and $f: A \longrightarrow M$ is given. Now we take pushout, the lowersequence splits and hence M is τ –pure injective.

(ii) $\Rightarrow$ (iii). This statement follows because $\mathrm{Ext}(F, M) = \mathrm{Pext}_{\tau}(F, M)$ for all $F \in \tau_1$.

(iii) $\Rightarrow$ (ii). Again, if $\mathrm{Ext}(F, M) = 0$ then the sequence $0 \longrightarrow M \longrightarrow N \longrightarrow F \longrightarrow 0$ splits for all $F \in \tau_1$.

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & \sigma(N) & = & \sigma(N) & & \\ & & & & \downarrow & & \downarrow & & N \\ 0 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0 \ldots\ldots\ldots (1) \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \rightleftarrows & P/\sigma(N) & \longrightarrow & N/\sigma(N) & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Where, $u: M \longrightarrow P, \pi: P \longrightarrow N, \lambda: N \longrightarrow \frac{N}{\sigma(N)}, \lambda': P \longrightarrow \frac{P}{\sigma(N)}, \mu: M \longrightarrow K$, $\mu': K \longrightarrow M.\ i: \sigma(N) \longrightarrow N, j: \sigma(N) \longrightarrow P$ and $q: P/\sigma(N) \longrightarrow K$.

It is given that the sequence (1) is τ –pure and we've $j: \sigma(N) \longrightarrow P$ which is a monomorphism. Now, $N/\sigma(N) \in \tau_1$ and hence, the right perpendicular sequence $0 \longrightarrow \sigma(N) \longrightarrow N \longrightarrow N/\sigma(N) \longrightarrow 0$ is τ –pure and hence, $\pi' \in \pi^{-1}(\tau)$ and so the epimorphism π' splits. Now we considering the perpendicular exact sequence, the identity map above sureties that the square is a pullback which in turn guarantees that μ is an isomorphism again if $\mu' = \mu^{-1}$, then $\mu' o(qo\lambda')o\, u = (\mu' oq)o(u' o\, \mu) = \mu' o\mu = 1$ and hence, the upper sequence splits also and so, $\mathrm{Pext}_{\tau}(N, M) = 0$ for all R –modules N.

(iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) It follows from the previous proposition by taking $\vartheta = \tau_1$.

Note 1.20

τ_1 – purity arises in the hypothesis of torsion free covers (M. L. Teply and J. S. Golan [18]).

Proposition 1.21

If for any module M, $M/\sigma(M)$ is projective, also M is τ –pure projective. Again, for every τ –pure projective module M, $M/\sigma(M)$ is a projective module handed every torsion free module is a factor of a projective torsion free module.

Conclusion

In this paper we consider an another notion of purity stronger than the Cohn's purity [13]. If $\mathcal{FG}$ denotes the class of all finitely generated R –modules. Since, this class is closed under factors. We shall give some characterizations of $\mathcal{FG}$ –purity and to determine its relationship with the $\mathcal{FG}$ –flat modules. We relativize this concept and also relate it with that of finite projectivity of Azumaya [10] with reference to a torsion theory and to study the inter-relationship between these concepts. We also consider finite σ –projectivity or (FG, σ) – pure flatness, cyclically σ – pure projectivity and cyclically σ – pure flatness, the concept of locally σ – projectivity and locally σ – splitness and study its inter- relationship with (FG, σ) – purity and semi-simple module. These relationships are very use-full for further its related works in ring and modules.

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