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Research Article

Propagation of Nonlinear Pulses in Long Lines with Dispersion in the Presence of Low Frequency Current and Voltage Fluctuations

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Abstract

The propagation of pulses of waves of current and voltage in long lines with nonlinearity and dispersion in the presence of low-frequency fluctuations of current and voltage, which can be caused by the influence of external EM fields generated by nearby electrical equipment of the industrial or energy object, is analytically investigated. It is shown that even neglecting losses, under the influence of low-frequency fluctuations external to the pulse, it is deformed, and, with propagation, the characteristic size of the pulse along the line and its amplitude change as $t^{3/2}$ and $t^{-3/2}$, respectively.

Keywords: Nonlinear Pulses; Voltage; Fluctuation

Basic equations and statement of the problem

Propagation in long lines with distributed parameters of nonlinear pulses of current and voltage waves (CVW), excited by external sources of the electromagnetic (EM) field, was studied theoretically and numerically in [1,2]. At this, in [1], for inhomogeneous telegraph equations describing CVW in lines with a linear load, exact analytical solutions were obtained for such sources as a remote lightning discharge, which induces CVW due to the spreading of charges, "pulled up" by the electrostatic field of a thunderstorm cloud. For lines with a nonlinear load, the resulting set of inhomogeneous Korteweg-de Vries (KdV) equations was solved numerically using the methods of numerical integration of nonlinear systems developed in [2,3]. However, in both cases, it was assumed that low-frequency (in comparison with the characteristic pulse length) oscillations of the current and voltage in the line were initially absent. In reality, almost always we meet in practice with a situation when signals propagating in lines (for example, in control networks) are influenced by external EM fields generated by nearby electrical equipment of an industrial or energy facility, the total effect of which onto the line has a pronounced chaotical character. As a result, such an effect can, in a fairly good approximation, be considered as stochastic fluctuations of the current (and, accordingly, voltage) induced in the line, the frequencies of which (on the order of the power frequency) are significantly lower than the characteristic frequencies of the CVW control pulses propagating through the network.

In this work, we study the problem of propagation of CVW pulses propagating in long lines that include nonlinear elements (for example, semiconductor (parametric) diodes, varistors or spark gaps as nonlinear capacitors, etc.) with a dispersion determined by the presence of inductive elements in the line, in presence of low-frequency fluctuations of current and voltage. For a line whose element is shown in figure 1, the equations describing the propagation of CVW have the form of the set of the KdV equations that take into account possible losses in the line [2]:

$$\partial_t I + (\alpha_1 I\,\partial_x U + \beta_1 \partial_x^3 U + RI)/L = 0,$$
$$\partial_t U + (\alpha_2 U\,\partial_x I + \beta_2 \partial_x^3 I + GU)/C = 0, \text{ ----(1)}$$
$$-\infty < x < \infty, \quad t > 0.$$

Figure 1: Element of long line with magnetic coupling.

Here *R, C, L, G* are distributed parameters: resistance, capacitance, inductance and leakage coefficient (conductance), calculated per unit length; and α_1, α_2 and β_1, β_2 are parameters that determine the contribution of nonlinear and dispersion effects, respectively. Following [3], we rewrite eqs. (1) in dimensionless form:

$$\begin{cases} \partial_t I + \widetilde{\alpha}_1 I \partial_x U + \widetilde{\beta}_1 \partial_x^3 U + \widetilde{\gamma}_1 I = 0, \\ \partial_t U + \widetilde{\alpha}_2 U \partial_x I + \widetilde{\beta}_2 \partial_x^3 I + \widetilde{\gamma}_2 U = 0, \end{cases} \text{ -------- (2)}$$

where coefficients $\widetilde{\alpha}_{1,2}$, $\widetilde{\beta}_{1,2}$, $\widetilde{\gamma}_{1,2}$ are also dimensionless, and for the analysis simplification let us first assume that the line losses are negligible: $\widetilde{\gamma}_{1,2} \approx 0$, since their presence can "mask" the effects caused by the action of the stochastic current and voltage fluctuations on propagation of the CVW pulse in the study (we will discuss the effect of losses in the final part of the paper).

Consider one (any) of the equations obtained in this way and, omitting the "tildes" and indices at the coefficients, and also assuming, for example, for the first equation of set (2) that at times much less than the characteristic period of fluctuations, the current and voltage are linearly related by the relation *U = IR*, we write:

$$\partial_t U + \alpha U \partial_x U + \beta \partial_x^3 U = 0,$$

where $\alpha = \widetilde{\alpha}_1$, $\beta = \widetilde{\beta}_1 R$.

For simplicity, we make change $U \to -(6/\alpha)u$ and introduce into the equation a term describing stochastic fluctuations:

$$\partial_t u - 6u\partial_x u + \partial_x^3 u - \eta(t) = 0. \text{ -------- (3)}$$

Equation (3) is the so-called "stochastic" KdV equation, first studied (regardless of the type of medium) by M. Wadati [4]. It is known from [3,5] that this equation for $\eta(t) = 0$ describes the evolution of nonlinear waves and solitons in a wide variety of dispersive media. For definiteness, we will investigate the effect of stochastic oscillations of the current and voltage in the line on the soliton of the KdV equation, since at $\eta(t) = 0$ it is a stable formation and propagates without changing its shape and velocity, although the approach carried out below is quite general (for example, in Sec. 2, the problem is solved for waves of any type described by the equation in the general setting).

In eq. (3), $\eta(t)$ describes the external "noise" when the characteristic dimensions of the CVW soliton l_s are much less than the coherent length of the noise l_n. This is a particular case of a more general one, when the external noise is described by a term of the form $\eta(x,t)$. However, being simpler for analytical study, the considered particular case allows us to obtain an exact result and gives us information that is very useful for a more general situation when $l_s \gtrless l_n$.

General exact solution

First of all, note that Eq. (1) is related to the KdV equation

$$\partial_t V - 6V\partial_\xi V + \partial_\xi^3 V = 0$$

by the Galilean transformation

$$u(t,x) = V(t,\xi) + W(t), \quad W(t) = \int_0^t \eta(t)dt, \text{ ------ (4)}$$

$$\xi = x + m(t), \quad m(t) = 6\int_0^t W(t)dt$$

and, therefore, is a fully integrable system and can be integrated by the IST method [6]. Following the analysis performed in [4], we will assume that the external noise η(t) is Gaussian

$$\langle \eta(t_1)\eta(t_2)\ldots\eta(t_n)\rangle = 0 \quad \text{(нечетные } n\text{);}$$
$$= \Sigma\Pi\langle \eta(t_i)\eta(t_j)\rangle \quad \text{(четные } n\text{)} \quad \text{--------(5)}$$

and white $\langle \eta(t)\eta(t')\rangle = 2\varepsilon\delta(t-t')$. Here, angle brackets $\langle\ \rangle$ denote statistical averaging, and symbols $\Sigma\Pi$, as in [4], mean that we choose $n/2$ pairs (t_i, t_j), multiply $n/2$ times $\langle \eta(t_i)\eta(t_j)\rangle$ and sum over all different (n-1)!! In this case for $W(t)$ we have

$$\langle W(t)\rangle = 0, \qquad \langle W(t_1)W(t_2)\rangle = 2\varepsilon \min(t_1,t_2),$$
$$\langle \exp\left[cW(t)\right]\rangle = \exp\left[\tfrac{1}{2}c^2\langle W^2(t)\rangle\right], \qquad c = \text{const.} \text{ ----- (6)}$$

Let us first consider the problem in its most general setting. Let the functional of V (t, ξ) has the following form:

$$F\left[V(t,\xi)\right] = F\left[V(t,\xi), \partial_\xi V(t,\xi),\ldots\right] = F(t,\xi). \text{ ------ (7)}$$

Considering Fourier transform

$$F(t,\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\hat{F}(t,k)\,e^{ikx}, \quad \hat{F}(t,k) = \int_{-\infty}^{\infty} dx F(t,\xi)\,e^{-ikx},$$

with due account fluctuations of the coordinate ξ, we obtain [3]

$$\hat{F}(t,k) = \hat{F}_0(t,k)\exp\left[ik\,m(t)\right], \text{ ----- (8)}$$

where

$$\hat{F}_0(t,k) = \hat{F}(t,k)\Big|_{m=0} = \int_{-\infty}^{\infty} dx F(t,x)\,e^{-ikx} \text{ . ------ (9)}$$

Averaging statistically, we have

$$\langle \hat{F}(t,k)\rangle = \hat{F}_0(t,k)\,\hat{G}(k), \text{ ---------- (10)}$$

where for $\hat{G}(k) = \langle \exp[ikm(t)]\rangle$, using (5) and (6), we can write

$$\hat{G}(k) = \exp\left[-\tfrac{1}{2}k^2\langle m^2(t)\rangle\right], \quad \langle m^2(t)\rangle = 24\varepsilon t^3, \quad t>0 \text{ . -------- (11)}$$

Equation (10) shows that the averaged spectrum (8) of the functional $F[V(t,\xi,y)]$ is the product $\hat{F}(t,k,y)$ in the absence of noise (9) and Gaussian distribution (11). Thus, we have

$$\langle F[V(t,\xi)]\rangle = \langle F(t,\xi)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\hat{F}_0(t,k)\,\hat{G}(k)\,e^{ikx} \text{ . ---------- (12)}$$

Using the convolution theorem, one can also obtain from solution (12) [4]

$$\langle F[V(t,\xi)]\rangle = \int_{-\infty}^{\infty} ds F[V(t,s)]G(x-s) \text{, ------------ (13)}$$

where

$$G(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\hat{G}(k)e^{iks} = \left[2\pi\langle m^2(t)\rangle\right]^{-1/2}\exp\left[-s^2/2\langle m^2(t)\rangle\right].$$

The obtained expressions (12) and (13) can now be used to study the dynamic behavior of solitons of Eq. (3), which we will now consider.

Dynamics of the CVW solitons

As an example, let us investigate the case when $F[V(t,\xi)]$ (7) is the functional of the one-soliton solution of Eq. (3). Calculating $\hat{F}_0(t,k)$ and $\hat{G}(k)$ using formulas (9) and (11), one can easily find $\langle \hat{F}(t,k)\rangle$ and then obtain $\langle u(t,x)\rangle$.

We will use, however, a more visual and simple method proposed in [4]. Consider the solution

$$V(t,x) = -2\nu^2\,\text{sech}^2\left[\nu(x-x_0) - 4\nu^3 t\right] \text{ ---------- (14)}$$

(ν is constant, which has the meaning of the eigenvalue corresponding to the soliton – see [6]) and, taking into account the change $\xi = x + m(t)$ and formula (4), we write the solution in the form

$$u(t,x) = W(t) - 2\nu^2\,\text{sech}^2\left[\nu(x-x_0) - 4\nu^3 t + 6\nu\int_0^t W(t')dt'\right].$$

Further, taking the statistical average and using formulas (5), (6), we obtain:

$$\langle u(t,x)\rangle = -2\nu^2\left\langle \text{sech}^2\left[\nu(x-x_0) - 4\nu^3 t + 6\nu\int_0^t W(t')dt'\right]\right\rangle =$$
$$= 8\nu^2\sum_{n=1}^{\infty}(-1)^n n\left\langle \exp\left[2n\left\{\nu(x-x_0) - 4\nu^3 t + 6\nu\int_0^t W(t')dt'\right\}\right]\right\rangle.$$

Second and third relations (6) give [3]

$$\left\langle \exp\left\{12nk\int_0^t W(t')dt'\right\}\right\rangle =$$
$$\exp\left\{\tfrac{1}{2}(12nk)^2\int_0^t dt_1\int_0^t dt_2\langle W(t_1)W(t_2)\rangle\right\} =$$
$$= \exp\left(48n^2\nu^2\varepsilon t^3\right), \qquad t>0.$$

Thus, we have

$$\langle u(t,x)\rangle = 8\nu^2 \sum_{n=1}^{\infty} (-1)^n n\, e^{na+n^2 b}, \text{ ------- (15)}$$

where

$$a = 2\left[\nu(x-x_0) - 4\nu^3 t\right],\ b = 48\nu^2 \varepsilon t^3. \text{ ----------- (16)}$$

Note that formula (15) was obtained under the assumption that the “noise” is Gaussian. For “noise” that is not white, the expression for the parameter b (16) will be more complicated. Following [3], we obtain from (15)

$$\partial_b \langle u(t,x)\rangle = \partial_a^2 \langle u(t,x)\rangle,$$

$$\langle u(t,x)\rangle\big|_{b=0} = -2\nu^2 \operatorname{sech}^2(a/2). \text{ --------- (17)}$$

It follows from the first equality in (17) that the dynamic behavior of the soliton of the stochastic KdV equation is described by the diffusion equation, where the role of time is played by the parameter b, and the role of the spatial coordinate is played by a. Note that eq. (15) can be written in the form of the Fourier transform, and then the solution of eq. (17) takes the form

$$\langle u(t,x)\rangle = -8\nu^2 \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\pi k}{\sinh \pi k} e^{-bk^2} e^{iak} \text{ . -------- (18)}$$

Formula (18) gives a spectral representation of the solution of the stochastic KdV equation in the presence of stochastic Gaussian fluctuations of the wave field, at this the Fourier transform of the statistical mean $\langle u(t,x)\rangle$ is the product of the purely soliton part $-8\nu^2 \pi k / \sinh \pi k$ and the diffusion part $\exp(-bk^2)$. Using the convolution theorem, solution (18) can be rewritten in the following form [5]:

$$\langle u(t,x)\rangle = -\frac{\nu^2}{\sqrt{\pi b}} \int_{-\infty}^{\infty} ds \operatorname{sech}^2(s/2) \exp\left[-(a-s)^2/4b\right].$$

Based on the result (18), let us now consider the dynamic behavior of a soliton in the presence of a Gaussian “noise”. According to [5], from (18) one can obtain:

a) at $b \equiv 48\nu^2 \varepsilon t^3 < 1$

$$\langle u(t,x)\rangle = -2\nu^2 \sum_{n=0}^{\infty} \frac{1}{n!} b^n \frac{\partial^{2n}}{\partial a^{2n}} \operatorname{sech}^2(a/2), \text{ --------- (19a)}$$

b) at $b > 1$

$$\langle u(t,x)\rangle = -\frac{4\nu^2}{\sqrt{\pi}} \left(1 + \sum_{n=1}^{\infty} \frac{(2^{2n}-2)B_n \pi^{2n}}{(2n)!} \frac{\partial^n}{\partial b^n}\right) \frac{1}{\sqrt{b}} e^{-a^2/4b}, \text{ (196)}$$

where B_n are the Bernoulli numbers. Expressions (19) show that at $t=0$ $\langle u(t,x)\rangle$ is defined by the right-hand side of formula (14) with $t=0$, and at $t \to \infty$

$$\langle u(t,x)\rangle = -\frac{\nu}{\sqrt{3\pi\varepsilon}} t^{-3/2} \exp\left[-\frac{(x-x_0-4\nu^2 t)^2}{48\varepsilon t^3}\right].$$

From the last expression it can be seen that during evolution as a result of the action of external “noise” the soliton is deformed, and, asymptotically, its characteristic size along the direction of propagation and the amplitude change, respectively, as $t^{3/2}$ and $t^{-3/2}$, that is not a consequence of diffusion or dissipative effects that could associated with losses, since the region occupied by the soliton is invariant, i.e. the integral $\int_{-\infty}^{\infty} \langle u(t,x)\rangle\, dx$ is conserved. This can be easily verified by calculation:

$$\int_{-\infty}^{\infty} \langle u(t,x)\rangle\, dx = -8\nu^2 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\pi k}{\sinh \pi k} e^{-bk^2} e^{iak} =$$

$$= -8\nu^2 \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\pi k}{\sinh \pi k} e^{-bk^2} 2\pi\delta(2\nu k) = -4\nu.$$

Conclusion

In conclusion, we note that we considered the influence on the structure and evolution of the soliton of the KdV equation, describing the CVW pulse, of the stochastic oscillations present in the system (in our case, in the line), in form of the external white “noise” η (t). In a more general case, the KdV equation can take the form [4]

$$\partial_t u - 6u\partial_x u + \partial_x^3 u + \gamma u - \eta(t,x) = 0, \text{ ------------ (20)}$$

(where fourth term – compare (20) with the eqs. (2) – describes the losses in the line), however, the analysis performed remains valid when the characteristic time $t_s < 1/\gamma$ and the characteristic size of the soliton $l_s < l_n$ (l_n is the coherent "length" of the noise). In the case when $l_s \sim l_n$, the Galilean transformation (4) turns out to be incorrect and it is necessary to generalize the method of the inverse scattering problem, as it was done, for example, for the KdV equation in [7,8]. To obtain exact (analytical) solutions of eq. (20) is not possible, and the only way to study the dynamics of its solutions is numerical integration, which can be successfully carried out using the methods developed in [2,3,5].

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